On embeddings of gauge groups in Yang-Mills theory

Meyers, C.; Roo, M. de; Sorba, P.

Published in:
Nuclear Physics B

DOI:
10.1016/0550-3213(78)90012-3

Nuclear Physics B140 (1978) 533–546

ON EMBEDDINGS OF GAUGE GROUPS IN YANG-MILLS THEORY

C. MEYERS, M. de ROO and P. SORBA
CERN, Geneva, Switzerland

Received 10 March 1978
(Revised 9 May 1978)

We discuss some group theoretical properties of Yang-Mills theories. We consider conditions necessary and sufficient to decide if the gauge field is reducible and prove some related theorems. We give criteria for embeddings and work out the case of SU(3) explicitly.

1. Introduction

Nowadays it is widely accepted that classical solutions of Yang-Mills theories are of physical interest. For a recent review of this subject see, for example, ref. [1]. Belavin et al. [2] have given a self-dual solution of the SU(2) Yang-Mills theory. The properties of this solution have been extensively studied, and other solutions have been obtained. However, the construction of explicit solutions has not gone beyond the SU(2) gauge group. All solutions which are known to us are either SU(2) solutions, or embeddings of these SU(2) solutions in some larger gauge group. Nevertheless, one of the most interesting physical cases is the SU(3) colour gauge group of quantum chromodynamics. Attempts to find non-trivial solutions for larger groups (non-trivial in the sense that they are not embeddings of SU(2) or SU(2) × SU(2)) have not produced any result so far, although it is not easy to see that a solution corresponds in fact to an embedding [3–5] (see the remark at the end of sect. 5).

The existence of self-dual field configurations in SU(N) Yang-Mills theory, which are non-trivial in the sense defined above, has been established [6,7]. It can therefore be expected that the search for such solutions will continue. In this respect, the Bäcklund-type transformations which have recently been developed for SU(2) and SU(N) [8] may be helpful. However, it is not clear if this method, when applied to an embedding of an SU(2) solution in SU(N), will produce a non-trivial SU(N) solution of finite action.

It is not always easy to conclude that a given solution of an SU(N) theory is in fact an embedding of a smaller group. Clearly a solution which in one gauge is obviously of SU(2) type in SU(N) can be very complicated after a sufficiently tricky

SU(N) gauge transformation. This is true in particular for the algebraic structure of the gauge potential $A_\mu(x)$, since a gauge transformation does not act simply on A_μ as a conjugation by a group element. It is therefore of interest to have criteria which allow one to distinguish between embeddings and non-trivial solutions, especially if these criteria are sufficiently simple to be used in explicit algebraic calculations.

In this paper we answer some of the questions related to embeddings. In sect. 2 we discuss some general properties of gauge fields and embeddings, we give some definitions and establish our notation. In sect. 3 we give a theorem which provides us with a local characterization of an embedding. We show that if the field strength tensor and a number of its covariant gauge derivatives belong, in a neighbourhood of some point x, to a subalgebra $\mathcal{H}$ of an algebra $\mathcal{G}$, then there exists a gauge in which also the gauge potential $A_\mu(x)$ belongs to $\mathcal{H}$, in this same neighbourhood. We must then establish that such a local property is, in fact, sufficient to characterize an embedding globally. This we demonstrate in sect. 4, where we discuss a number of global properties of gauge fields. In sect. 5 we consider explicitly SU(3), and give criteria which allow one to recognize an SU(2) or O(3) embedding in SU(3). These criteria are again of a local nature, but our results in sect. 4 show that they are nevertheless sufficient. Some technical details of the proofs we give in sects. 3 and 4 have been gathered in the appendices.

2. General properties of gauge fields and embeddings

In this section we shall discuss some aspects of gauge theories which are related to the problem of embeddings. Let us first establish some notations.

In order to define a gauge structure on the compactified Euclidean space E^4 ($\cong S^4$) one must first cover E^4 with a finite number of overlapping open regions O_i (for S^4 two regions are sufficient). Then the gauge potential $A_\mu(x)$ is defined in each of the regions (gauge patches) O_i by analytic functions:

$$A^i_\mu(x) = A^{i\alpha}_\mu(x) X_\alpha \ , \tag{2.1}$$

where the X_α are the generators of the gauge group G satisfying

$$[X_\alpha, X_\beta] = 2i f_{\alpha\beta\gamma} X_\gamma , \tag{2.2}$$

with real structure constants $f_{\alpha\beta\gamma}$. In the regions of overlap $O_i \cap O_j$ there exist transition functions $U_{ij}(x)$ of G, regular in $O_i \cap O_j$, which relate $A^i_\mu(x)$ and $A^j_\mu(x)$:

$$A^i_\mu(x) = U_{ij}(x) A^j_\mu(x) U^{-1}_{ij}(x) + iU_{ij}(x)\, \partial_\mu U^{-1}_{ij}(x) \ . \tag{2.3}$$

The field strength tensor, which also takes its value in the Lie algebra $\mathcal{G}$ of G, is given by

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - i[A_\mu(x), A_\nu(x)] \ , \tag{2.4}$$

where we have suppressed the index labelling the patches O_i. We shall also need gauge derivatives of $F_{\mu\nu}(x)$, given by

$$D_\rho F_{\mu\nu}(x) = \partial_\rho F_{\mu\nu}(x) - i[A_\rho(x), F_{\mu\nu}(x)] \,,$$
$$D^2_{\rho\sigma} F_{\mu\nu}(x) = \partial_\rho D_\sigma F_{\mu\nu}(x) - i[A_\rho(x), D_\sigma F_{\mu\nu}(x)] \,, \tag{2.5}$$

etc. Under a gauge transformation $U(x)$, where U is an element of G, we have the following transformation properties:

$$A'_\mu(x) = U(x)\, A_\mu(x)\, U^{-1}(x) + iU(x)\, \partial_\mu U^{-1}(x) \,,$$
$$F'_{\mu\nu}(x) = U(x)\, F_{\mu\nu}(x)\, U^{-1}(x) \,,$$
$$D'_\rho F'_{\mu\nu}(x) = U(x)\, D_\rho F_{\mu\nu}(x) U^{-1}(x) \,, \tag{2.6}$$

etc. Let us now define what we mean by an embedding. A gauge potential $A_\mu(x)$ of a group G is an embedding if there exists a gauge in which the $A^i_\mu(x)$ for all the regions O_i belong to a subalgebra $\mathcal{H}$ of the algebra $\mathcal{G}$ of G, and the transition functions belong to the group H generated by $\mathcal{H}$.

Clearly, once the gauge potential is in $\mathcal{H}$, we can still make gauge transformations of the group H, and in this sense we can consider the theory to be an H gauge theory. The information that we started out with a larger group G is not completely lost, certain quantities (for example, the topological charges) still depend on the way the group H has been embedded in G [9,15]. The equations of motion recognize only the group H.

It will be useful in what follows to define also a local kind of reducibility: we shall call $A^i_\mu(x)$ reducible in O_i if there exists a gauge transformation $U(x)$, defined and regular in O_i, which transforms $A^i_\mu(x)$ into a subalgebra $\mathcal{H}$ of G. $A^i_\mu(x)$ is irreducible in O_i if it cannot be transformed into a smaller subalgebra than the last one obtained by reduction.

To study certain group theoretical properties of a gauge theory, it is, because of the transformation properties (2.6), advantageous to consider properties of the field strength tensor. The question then arises to what extent the knowledge of the field strength tensor suffices to determine the field A_μ. In recent years much attention has been paid to this reconstruction problem. For classical electrodynamics (i.e., the U(1) gauge theory), the field strength completely determines the potential A_μ, if one considers a simply connected region. For non-Abelian theories it is known that the field strength tensor alone is not sufficient: a counter-example has been constructed by Wu and Yang [10]. They give two gauge fields A_μ, for an SU(2) gauge theory, which cannot be gauge transformed into each other, but which nevertheless give the same $F_{\mu\nu}$. So, for non-Abelian theories, more information has to be provided to characterize the gauge field. A general theorem has been given by Gu and Yang [11]. They prove that for a G gauge field theory the gauge field can be determined by the field strength and its gauge derivatives up to pth order, where the integer p is at most the order of the group G. By "determined" they mean of course up to a gauge transformation which leaves $F_{\mu\nu}, D_\rho F_{\mu\nu}, \dots$ the same. This theorem is useful for our purposes, since it characterizes the gauge field, up to a certain class of gauge transformations, by quantities which transform in the adjoint representation by conjugation (2.6), and which are therefore eminently suitable for investigation by group theoretical methods. As an application of the theorem consider the case that $F_{\mu\nu}(x) \equiv 0$. As a consequence all gauge derivatives of F also vanish. Since $A_\mu(x) = 0$ is a gauge potential corresponding to $F_{\mu\nu} = 0$, the theorem tells us that the most general potential is a pure gauge, i.e.,

$$A_\mu(x) = iU(x)\,\partial_\mu U^{-1}(x)\,. \tag{2.7}$$

Given the Gu-Yang theorem, one can now ask a question about embeddings. Suppose that $F_{\mu\nu}$ and its gauge derivatives up to the order required by the theorem, all belong to a subalgebra $\mathscr{H}$ of the algebra $\mathscr{G}$, and we know that $A_\mu \in \mathscr{G}$ exists, can we draw the conclusion that by a gauge transformation U of G, which commutes with F and its derivatives, A_μ can be transformed into $A'_\mu \in \mathscr{H}$? Clearly this is not a direct consequence of the Gu-Yang theorem, since the existence of one A_μ in $\mathscr{H}$ would be required for its application. In sects. 3 and 4 we shall demonstrate this extension of the Gu-Yang theorem with group theoretical methods. We believe the proof to be instructive since it clearly shows the restrictions on the freedom of A_μ to take values outside the subalgebra $\mathscr{H}$. The understanding of these restrictions helps us to formulate, in sect. 5, practical criteria which distinguish between embeddings and solutions of the full gauge group.

The problem of recognizing embeddings would be much simplified if one could find a gauge condition which restricts A_μ in such a way that the true gauge group is immediately visible. An example of such a gauge condition can be found in a recent paper by Bernard et al. [12]. However, such gauges cannot be imposed everywhere on the compactified Euclidean space S^4, and therefore do not solve the problem globally. Also, the local construction of such a gauge involves solving differential equations, and does not seem to be very practical.

We cannot exclude the possibility that our theorems could also be established by a differential geometrical approach (the holonomy group). For a recent review of the relation between differential geometry and Yang-Mills theory we refer to the lectures by Stora [13] and references therein.

3. Characterization of embeddings

In this section we state and prove the theorem mentioned in sect. 2, for a gauge field defined on an open set of E^4 (one patch). In sect. 4 we will discuss global properties.

Theorem

Let $\mathscr{F}$ be a G gauge field, where G is a compact group, on a simply connected open set of E^4 with an analytic gauge potential $A_\mu(x)$. If the field strength $F_{\mu\nu}(x)$

and its covariant gauge derivatives $D_\rho F_{\mu\nu}, D^2_{\rho\sigma} F_{\mu\nu}, \ldots$ can be written in terms of a proper subalgebra $\mathcal{M}$ of $\mathcal{G}$, the Lie-algebra of G, then there exists a gauge in which $A_\mu(x)$ can be written in terms of $\mathcal{M}$, i.e., the field $\mathcal{F}$ is an embedding of the subgroup M of G in this open set.

We first construct the algebras $\mathcal{H}_x$, which are generated by $F_{\mu\nu}(x), D_\rho F_{\mu\nu}(x) \ldots$. Since G is compact, these algebras $\mathcal{H}_x$ must be of the form

$$\mathcal{H}_x = \mathcal{S}_x \oplus \mathcal{A}_x ,$$

where $\mathcal{S}_x$ is semi-simple and $\mathcal{A}_x$ Abelian. The algebra of a compact group has only a finite number of semi-simple subalgebras (up to conjugations). Let us call them $\mathcal{S}_1$, $\mathcal{S}_2, \ldots, \mathcal{S}_n$ and choose standard, and of course x-independent, bases for them. This implies that there exist group elements $U(x)$, with

$$\mathcal{S}_x = U(x)\, \mathcal{S}_{i(x)} U(x)^{-1} ,$$

where $\mathcal{S}_{i(x)}$ is one of the standard subalgebras. Since $\mathcal{H}_x$ is spanned by vectors which depend analytically on the variable x, the dimension and the structure constants of $\mathcal{H}_x$ must be independent of x almost everywhere. This is also true for $\mathcal{S}_x$, and therefore $\mathcal{S}_{i(x)}$ must be the same almost everywhere:

$$\mathcal{S}_x = U(x)\, \mathcal{S}_{i_0} U(x)^{-1} .$$

We then perform the analytic gauge transformation $U^{-1}(x)$, and find that $F'_{\mu\nu}(x)$, $D'_\rho F'_{\mu\nu}(x), \ldots$ generate the algebra

$$\mathcal{H}_x = \mathcal{S} \oplus \mathcal{A}'_x .$$

In the following we drop the primes and take the semi-simple part of $\mathcal{H}_x$ to be independent of x. We must now choose a basis for the Cartan subalgebra $\mathcal{C}$ of $\mathcal{G}$. This we do by taking the basis of the Cartan algebra of $\mathcal{S}$ first, and completing to find a basis for $\mathcal{C}$. We can assume without loss of generality that the Abelian part of $\mathcal{H}_x$ is in $\mathcal{C}$, and that the x-dependent generators of $\mathcal{A}_x$ can be written as

$$a_i(x) = \varphi_{ij}(x)\, c_j ,$$

where the c_j are generators of $\mathcal{C}$, and the analytic functions $\varphi_{ij}(x)$ for fixed i are linearly independent. The algebra generated by the c_j appearing in the $a_j(x)$ we shall call $\mathcal{A}$. It is the smallest Abelian algebra containing all the $\mathcal{A}_x$ for different x. Therefore

$$\mathcal{H} = \mathcal{S} \oplus \mathcal{A} \supset \mathcal{H}_x = \mathcal{S} \oplus \mathcal{A}_x .$$

Clearly $\mathcal{H}$ is a subalgebra of $\mathcal{M}$. We shall show that $A_\mu(x)$ can in fact be written in terms of $\mathcal{H}$.

Let us now define the algebra $\widetilde{\mathcal{H}}$, consisting of all elements of $\mathcal{G}$ which commute with $\mathcal{H}$. Similarly, we have $\widetilde{\mathcal{H}}_x$. We will show that

$$\widetilde{\mathcal{H}} = \widetilde{\mathcal{H}}_x .$$

Clearly we have the following relation

$$\widetilde{\mathcal{H}}_x = \widetilde{\mathfrak{J}} \cap \widetilde{\mathcal{A}}_x \supset \widetilde{\mathcal{H}} = \widetilde{\mathfrak{J}} \cap \widetilde{\mathcal{A}} \ ,$$

so that we only have to prove that $\widetilde{\mathcal{A}} = \widetilde{\mathcal{A}}_x$. We give this proof in appendix A. Now that we know that $\widetilde{\mathcal{H}} = \widetilde{\mathcal{H}}_x$ the rest fo the proof is simple. We decompose the algebra $\mathcal{G}$ into two parts, $\mathcal{H}$ and $\mathcal{H}^\perp$ which are orthogonal with respect to some invariant scalar product. Then

$$A_\mu(x) = A_\mu^{\mathcal{H}}(x) + A_\mu^\perp(x) \ .$$

Because $F_{\mu\nu}(x), D_\rho F_{\mu\nu}(x)$, ... are in $\mathcal{H}_x \subset \mathcal{H}$ we find with this decomposition and using (2.5) that

$$[A_\mu^\perp(x),\ \mathcal{H}_x] \subset \mathcal{H}.$$

But from the orthogonality of $\mathcal{H}$ and $\mathcal{H}^\perp$ follows that for any $h \in \mathcal{H}, h^\perp \in \mathcal{H}^\perp$,

$$[h, h^\perp] \in \mathcal{H}^\perp \ ,$$

and therefore

$$[A_\mu^\perp(x),\ \mathcal{H}_x] = 0 \ .$$

We know that $A_\mu^\perp(x)$ belongs to $\widetilde{\mathcal{H}}_x$, and therefore also to $\widetilde{\mathcal{H}}$. In fact, $A_\mu^\perp$ must belong to $\widetilde{\mathcal{H}} \cap \mathcal{H}^\perp$, which is again an algebra. Let us now calculate the component of $F_{\mu\nu}$ on $\widetilde{\mathcal{H}} \cap \mathcal{H}^\perp$, which of course must vanish:

$$F_{\mu\nu}^{\widetilde{\mathcal{H}} \cap \mathcal{H}^\perp} = \partial_\mu A_\nu^\perp - \partial_\nu A_\mu^\perp - i[A_\mu^\perp, A_\nu^\perp] = 0 \ .$$

Since the gauge potential corresponding to a vanishing field strength tensor is a pure gauge, we find that $A_\mu^\perp$ can be gauged away, which means that $A_\mu \in \mathcal{H} \subset \mathcal{M}$.

We have seen that in fact it is not the algebra $\mathcal{M}$ which is important, but the algebra generated by $F_{\mu\nu}(x)$, $D_\rho F_{\mu\nu}(x)$, ..., which we have called $\mathcal{H}_x$. We have seen that the semi-simple part of the true gauge group, H, can immediately be obtained by constructing $\mathcal{H}_x$ in an almost arbitrary point x, the points where a smaller algebra will be found being of measure zero in E^4. The remaining problem, and the reason for the rather lengthy proof, is caused by the possibility that the Abelian part of $\mathcal{H}_x$ depends in an essential way on x.

4. Global properties

In sect. 3 we have studied the conditions under which the gauge potential A_μ can be reduced in an open set $\bar{O}$ of S^4 to a proper subalgebra of $\mathcal{G}$. Since two open sets are needed to cover S^4, these conditions are not *a priori* sufficient to characterize an embedding. According to our definition in sect. 2, the gauge potentials must be reducible in both open sets to the same subalgebra $\mathcal{H}$ of $\mathcal{G}$, and the transition function which relates the gauge potentials must also be reducible to a proper subgroup of G.

In this section we shall show that in fact the reducibility of the gauge potential in one of the gauge patches is a surfficient condition for an embedding. To do this we must prove that if in $\mathcal{O}_1$ the gauge potential A^1_μ is reducible to $\mathcal{H}$, then A^2_μ in $\mathcal{O}_2$ is reducible to the same subalgebra, and that the transition function U is reducible to the associated subgroup. We shall first establish that A^1_μ and A^2_μ can be reduced to the same subalgebra $\mathcal{H}$. Of course we must use the fact that a transition function relating A^1_μ and A^2_μ exists. Afterwards we shall show that the transition function can be reduced to H.

So we consider two gauge potentials $A^1_\mu(x)$ and $A^2_\mu(x)$, defined in $\mathcal{O}_1$ and $\mathcal{O}_2$, respectively, by analytic functions $A^{1\alpha}_\mu(x)$, $A^{2\alpha}_\mu(x)$:

$$A^1_\mu(x) = A^{1\alpha}_\mu(x)\,\lambda^1_\alpha\,,$$
$$A^2_\mu(x) = A^{2\alpha}_\mu(x)\,\lambda^2_\alpha\,, \tag{4.1}$$

where λ^i_α are generators of subalgebras $\mathcal{H}_i$ of $\mathcal{G}$. The transition function $U(x)$ is defined and regular in $\mathcal{O}_1 \cap \mathcal{O}_2$ and transforms A^1_μ into A^2_μ:

$$A^2_\mu(x) = U(x)\,A^1_\mu(x)\,U^{-1}(x) + iU(x)\,\partial_\mu U^{-1}(x)\,. \tag{4.2}$$

We shall assume for the moment that A^1_μ and A^2_μ are irreducible with respect to $\mathcal{H}_1$ and $\mathcal{H}_2$, i.e., they have both been reduced to the smallest possible subalgebra. We shall then show that $\mathcal{H}_1$ and $\mathcal{H}_2$ must be the same, up to a constant (x-independent) conjugation.

Since it is the transformation $U(x)$ which relates $\mathcal{H}_1$ and $\mathcal{H}_2$, we must first establish some properties of this transition function. We know that in $\mathcal{O}_1 \cap \mathcal{O}_2$

$$F^2_{\mu\nu}(x) = U(x)\,F^1_{\mu\nu}(x)\,U^{-1}(x)\,,$$
$$(D_\rho F_{\mu\nu}(x))^2 = U(x)(D_\rho F_{\mu\nu}(x))^1\;U^{-1}(x)\,, \tag{4.3}$$

etc., where $F^i_{\mu\nu}(x)$, $(D_\rho F_{\mu\nu})^i(x)$, ..., belong to $\mathcal{H}_i$.

As in sect. 3, we define the algebras $\mathcal{H}_{1x}$ and $\mathcal{H}_{2x}$. We know from (4.3) that there exists a transformation $U(x)$ which transforms $\mathcal{H}_{1x}$ into $\mathcal{H}_{2x}$:

$$\mathcal{H}_{2x} = U(x)\;\mathcal{H}_{1x}\,U^{-1}(x)\,. \tag{4.4}$$

In appendix B we show that this is sufficient to conclude that the algebras $\mathcal{H}_{1x}$ and $\mathcal{H}_{2x}$ are the same, and therefore also $\mathcal{H}_1 = \mathcal{H}_2$, possibly after a constant (x-independent) gauge transformation has been performed.

So we can now conclude that if A^1_μ and A^2_μ are reduced to the smallest possible subalgebras $\mathcal{H}_1$ and $\mathcal{H}_2$, then $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$. What remains to be shown is that the transition function $U(x)$ can be chosen in the group H associated with $\mathcal{H}$. In appendix C, we show that the most general transformation $U(x)$, which satisfies (4.4) with $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, is of the form:

$$U(x) = U_{\mathrm{H}}(x)\,U_{\widetilde{\mathrm{H}}_\perp}(x)\,D\,, \tag{4.5}$$

where $U_{\mathrm{H}}(x)$ is a transformation of H, $U_{\widetilde{\mathrm{H}}_\perp}(x)$ is a transformation of the group $\widetilde{\mathrm{H}}_\perp$

associated with $\widetilde{\mathcal{H}}_\perp = \mathcal{H}^\perp \cap \widetilde{\mathcal{H}}$, and D is a constant transformation which has the further property that

$$DF^1_{\mu\nu}D^{-1} \in \mathcal{H} \;,$$
$$D(D_\rho F_{\mu\nu})^1\, D^{-1} \in \mathcal{H} \;, \tag{4.6}$$

etc. Since D is regular in O_1, we can redefine our gauge potential in O_1 to be

$$A^{1\prime}_\mu(x) = DA^1_\mu(x)\, D^{-1} \;, \tag{4.7}$$

and define a new transition function

$$U'(x) = U_{\mathrm{H}}(x)\, U_{\widetilde{\mathrm{H}}_\perp}(x) \;, \tag{4.8}$$

which transforms $A^{1\prime}_\mu(x)$ into $A^2_\mu(x)$:

$$A^2_\mu(x) = U_{\mathrm{H}}(x)\, U_{\widetilde{\mathrm{H}}_\perp}(x)\, A^{1\prime}_\mu(x)\, U^{-1}_{\widetilde{\mathrm{H}}_\perp}(x)\, U^{-1}_{\mathrm{H}}(x)$$
$$+\, iU_{\mathrm{H}}(x)\, U_{\widetilde{\mathrm{H}}_\perp}(x)\, \partial_\mu(U^{-1}_{\widetilde{\mathrm{H}}_\perp}(x)\, U^{-1}_{\mathrm{H}}(x)) \,. \tag{4.9}$$

Since $U_{\widetilde{\mathrm{H}}}$ leaves $\mathcal{H}$, and therefore $A^{1\prime}_\mu$ invariant, we can simplify (4.9):

$$A^2_\mu(x) = U_{\mathrm{H}}(x)\, A^{1\prime}_\mu(x)\, U^{-1}_{\mathrm{H}}(x) + iU_{\mathrm{H}}(x)\, \partial_\mu U^{-1}_{\mathrm{H}}(x) + iU_{\widetilde{\mathrm{H}}_\perp}(x)\, \partial_\mu U^{-1}_{\widetilde{\mathrm{H}}_\perp}(x) \,. \tag{4.10}$$

The last term in (4.10) belongs to the Lie algebra $\widetilde{\mathcal{H}}_\perp$, and since all other contributions belong to $\mathcal{H}$, this last term must vanish. Therefore we see that

$$A^2_\mu(x) = U_{\mathrm{H}}(x)\, A^{1\prime}_\mu(x)\, U^{-1}_{\mathrm{H}}(x) + iU_{\mathrm{H}}(x)\, \partial_\mu U^{-1}_{\mathrm{H}}(x) \;, \tag{4.11}$$

so that $A^{1\prime}_\mu$ and A^2_μ are indeed related by a transition function $U_{\mathrm{H}}(x)$, belonging to the group H.

We have assumed in this section that $A^1_\mu(x)$ and A^2_μ have already been reduced to the smallest possible subalgebras. If this were not the case, the algebras in terms of which A^1_μ and A^2_μ are expressed may of course differ. However, in the proof of the theorem in sect. 3 we have shown explicitly how the smallest algebra can be obtained. Therefore the restriction of irreducibility made at the beginning of this section is not essential.

We can therefore state that the conditions given in the theorem of sect. 3, which are necessary and sufficient for local reducibility, are also sufficient for global reducibility of the gauge field.

5. Criteria for SU(3)

In this section we shall explain how one can recognize embeddings of subalgebras of SU(3). The arguments which we shall give are of a local nature, i.e., they concern only the gauge field, the field strength tensor and the gauge derivatives (defined in sect. 2) in an arbitrarily small neighbouhood of some point x. In this section we shall not concern ourselves with the singularity structure or the gauge patch structure of the gauge fields, which has been discussed in sects. 2 and 4.

The case of SU(3) is relatively simple because the number of semi-simple subalgebras up to a conjugation is small. Any subalgebra is a subalgebra of one of the maximal subalgebras, and of these there are only two for SU(3), corresponding to the subgroups SU(2) $\times$ U(1) and O(3), respectively. Our criteria allow to decide if some given SU(3) gauge field is an embedding of one of the two maximal subalgebras. Once this question has been settled it is a trivial matter to go further down the chain of subalgebras, i.e., to check if the embedding is, in fact, SU(2) or U(1). As a basis of the SU(3) algebra we take the usual λ matrices of Gell-Mann, and we assume therefore that the gauge field and all derived quantities are given in the form of Hermitean traceless 3×3 matrices. We must also choose a suitable basis for the subalgebras. Taking the pair $\{\lambda_3, \lambda_8\}$ as the Cartan-subalgebra of SU(3), we take bases for the subalgebras which contain as many elements of the Cartan-subalgebra as possible. For SU(2) $\times$ U(1) we take $\{\lambda_1, \lambda_2, \lambda_3, \lambda_8\}$, for O(3) the basis will be $\{-2\lambda_3, \lambda_4 + \lambda_7, \lambda_5 + \lambda_6\}$. Of course, the basis $\{\lambda_2, \lambda_5, \lambda_7\}$, which is more often used for O(3), is conjugate to the basis we have chosen.

Our reasoning can be summarized as follows. Given a gauge field $A_\mu(x)$ of SU(3), we calculate $F_{\mu\nu}(x)$. We then diagonalize a non-vanishing linear combination of $F_{\mu\nu}$, let us say $F_{12}(x)$, by a unitary transformation $U(x)$, and we calculate $A'_\mu(x)$ in this new gauge. Then there are three possibilities.

(i) $F_{12}(x)$ has three different non-zero eigenvalues. In this case the embedding if any, must be SU(2) $\times$ U(1), and this can be decided immediately, since then the matrices $A'_\mu(x)$ must have a particular structure.

(ii) $F_{12}(x)$ has one zero eigenvalue. Then the embedding can be SU(2) $\times$ U(1), which can again be recognized by simply looking at $A'_\mu(x)$, or O(3). O(3) can be recognized by verifying some algebraic relations which the components of A'_μ must satisfy.

(iii) $F_{12}(x)$ has two equal eigenvalues. This turns out to be the most complicated case, and we shall treat it at the end of this section.

So let us now look in more detail at cases (i), (ii) and (iii), and let us assume that the gauge potential $A_\mu(x)$ is such that $F_{12}(x)$ is diagonal. If F_{12} has three different non-zero eigenvalues we cannot have an embedding of O(3). Eigenvalues of $F_{\mu\nu}$ are gauge-invariant, and since any element of the O(3) subalgebra has one zero eigenvalue, this must be true in any gauge. The gauge transformation which diagonalizes F_{12} is unique up to a discrete (x-independent) transformation, which interchanges the eigenvalues, if F_{12} has three different non-zero eigenvalues. Therefore the potential $A_\mu(x)$ is now determined up to one of these discrete transformations. If we let these discrete transformations act on the basic representation of SU(2) $\times$ U(1), we see that we get three possible block-like structures

$$\left[\begin{array}{cc|c} & & 0 \\ & & 0 \\ \hline 0 & 0 & \end{array}\right], \quad \left[\begin{array}{c|c|c} & 0 & \\ \hline 0 & & 0 \\ \hline & 0 & \end{array}\right], \quad \left[\begin{array}{c|cc} & 0 & 0 \\ \hline 0 & & \\ 0 & & \end{array}\right].$$

If $A_\mu(x)$ has one of these structures, we have an embedding of SU(2) $\times$ U(1). If the form of A_μ is not one of these three, the gauge field is really an SU(3) gauge field.

If $F_{12}(x)$ has one zero eigenvalue, we can, without loss of generality, assume that F_{12} is along λ_3. The gauge transformations which leave λ_3 invariant are conjugations generated by λ_8. If we have an embedding of SU(2) $\times$ U(1), we must have once again one of the three block-like forms for A_μ, since these forms are not changed by a gauge transformation generated by λ_8. If A_μ is not in one of these forms, we can still have an embedding of O(3). The remaining gauge freedom is given by the transformations

$$U(x) = \exp(i\theta(x)\sqrt{\tfrac{1}{3}}\lambda_8)\ . \tag{5.1}$$

The action of this transformation on the generators of O(3) is

$$\begin{aligned} U(x)(\lambda_4 + \lambda_7)\,U^{-1}(x) &= (\lambda_4 + \lambda_7)\cos\theta(x) - (\lambda_5 - \lambda_6)\sin\theta(x)\ , \\ U(x)(\lambda_5 + \lambda_6)\,U^{-1}(x) &= (\lambda_4 - \lambda_7)\sin\theta(x) + (\lambda_5 + \lambda_6)\cos\theta(x)\ , \end{aligned} \tag{5.2}$$

while λ_3 is of course unchanged. A gauge transformation (5.1) also generates a pure gauge term in A_μ which is along λ_8. So, once we have excluded the possibility SU(2) $\times$ U(1), we must do the following.

(a) Check that A_μ has no component along λ_1 and λ_2.

(b) Check that the relation between the components A^i_μ for $i = 4, 5, 6$ and 7 is such that the angle $\theta(x)$ exists:

$$\frac{A^4_\mu - A^7_\mu}{A^5_\mu + A^6_\mu} = -\frac{A^5_\mu - A^6_\mu}{A^4_\mu + A^7_\mu} = \tan\theta(x)\ , \qquad \text{for all } \mu\ . \tag{5.3}$$

(c) If the angle $\theta(x)$ exists and has been calculated by (5.3), the gauge transformation $U(x)^{-1}$, with U as in (5.1), must remove the λ_8 component. This concludes the case where $F_{12}(x)$ has one zero eigenvalue.

Now suppose that $F_{12}(x)$ has two equal eigenvalues. Without loss of generality we can assume that $F_{12}(x)$ is along λ_8. Now the remaining gauge freedom is an SU(2) group, and although we could proceed as before and find the action of this SU(2) on the three SU(2) $\times$ U(1) forms, this will become quite complicated. Fortunately, there is a simpler argument, based on the fact that

$$\det[\lambda_8, B] = 0 \tag{5.4}$$

for any B in the SU(3) Lie algebra. So if we calculate $[F_{12}(x), F_{\mu\nu}(x)]$ either all commutators vanish, in which case all $F_{\mu\nu}$ must belong to the basic representation $\{\lambda_1, \lambda_2, \lambda_3, \lambda_8\}$ of SU(2) $\times$ U(1), or one of the commutators does not vanish, and therefore has a zero eigenvalue. In this last case we go to the gauge where the non-vanishing commutator is along λ_3, and in this new gauge $A'_\mu(x)$ must, if the embedding is SU(2) $\times$ U(1), have one of the three possible basic forms. If all commutators vanish, and all $F_{\mu\nu}$ have two equal eigenvalues we are forced to calculate gauge derivatives. As soon as we find any object which does not have two equal eigenvalues, or does not commute with λ_8, we diagonalize it and consider A_μ in that new gauge. If all gauge derivatives belong to SU(2) × U(1), we use the theorem of sect. 3 to say that we must have an embedding of SU(2) × U(1).

This concludes the criteria for SU(3). In practice, the only calculation which is relatively complicated will be the construction of the gauge transformation which diagonalizes $F_{12}(x)$. Once this is done, the question of embedding can, except in some rather special cases treated above, be settled by inspection.

SU(3) solutions have been proposed by Yates [3] and by Wilczek [15]. The solution of Wilczek is by construction an embedding of O(3) into SU(3), and the author explains how inequivalent embeddings can affect the topological charge (see also ref. [9]). The solution of Yates is constructed starting from an ansatz with cylindrical symmetry. This ansatz is a generalization to SU(3) of the one used by Witten [16] to construct SU(2) multi-instanton solutions. Yates' ansatz contains, in principle, field configurations which do not correspond to embeddings. However, the requirement of self-duality leads to equations of which Yates obtained only a particular solution, which after a little algebra, can be shown to reduce to Witten's SU(2) solutions embedded in SU(3). This is because the only remaining SU(3) generators for this particular solution are $\{\lambda_2, \lambda_5, \lambda_7\}$, corresponding to the O(3) subalgebra of SU(3).

Thanks are due to G. Girardi for his participation in the early stage of this work. It is also a pleasure to thank R. Flume, W. Nahm, H. Römer and R. Stora for useful conversations. We also thank W. Nahm for reading the manuscript.

Note added

After completion of this work we received a preprint of a recent paper by Bais and Weldon [17] on SU(3) instantons, which allows an interesting application of our criteria. The authors find solutions starting from an ansatz similar to the one used by Yates [3]. Besides the embedding of Witten's SU(2) solution, they also obtain a field configuration which they claim is not an embedding.

To verify their statement, we have applied the criteria given in sect. 5 to their solution. Using the cylindrical symmetry of their ansatz we have diagonalized the particular linear combination of their $F_{\mu\nu}$: $r_a F_{0a}$. Since we find three different non-zero eigenvalues it is then sufficient to check the structure of the matrix A_μ in the new gauge. We find that their solution is indeed a non-trivial SU(3) solution.

Appendix A

Let us show that $\tilde{\mathscr{A}}_x = \tilde{\mathscr{A}}$. Suppose $\tilde{\mathscr{A}}_x \neq \tilde{\mathscr{A}}$. We already know that $\tilde{\mathscr{A}}_x \supset \tilde{\mathscr{A}}$, so we assume that there exists an element $h(x)$ such that

$$[h(x), \mathcal{A}_x] = 0 \,, \ [h(x), \mathcal{A}] \neq 0 \,.$$

This means that there is at least one generator of $\mathcal{A}$, let us say C_{i_0}, such that

$$[h(x), C_{i_0}] \neq 0 \,. \tag{A.1}$$

There is at least one generator of $\mathcal{A}_x$, say $a_{j_0}(x)$, in which C_{i_0} appears, so that

$$[h(x), a_{j_0}(x)] = [h(x), \sum_i \varphi_{j_0 i}(x)\, C_i] = 0 \,. \tag{A.2}$$

We can take $h(x)$ to have no component on $\mathcal{C}$. We then consider the Cartan decomposition of $\mathcal{G}$, with respect to $\mathcal{C}$, so that

$$h(x) = h_\alpha(x)\, E_\alpha \,, \tag{A.3}$$

where the generators E_α and the roots $r_i(\alpha)$ satisfy

$$[E_\alpha, C_i] = r_i(\alpha)\, E_\alpha \,. \tag{A.4}$$

We substitute (A.3) in (A.2) and use (A.4) to obtain

$$\sum_i h_\alpha(x)\, \varphi_{j_0 i}(x)\, r_i(\alpha) = 0 \,, \qquad \forall \alpha \,. \tag{A.5}$$

From (A.1) we find that there is at least one value α_0 such that

$$h_{\alpha_0}(x)\, r_{i_0}(\alpha_0) \neq 0 \,. \tag{A.6}$$

We then calculate (A.5) for the value α_0, and obtain a non-trivial linear relation between the functions $\varphi_{j_0 i}(x)$. This relation is non-trivial because in the sum over i both $\varphi_{j_0 i_0}(x)$ and $r_{i_0}(\alpha_0)$ do not vanish. This is in contradiction with the known properties of $\mathcal{A}_x$, and therefore

$$\tilde{\mathcal{A}}_x = \tilde{\mathcal{A}} \,.$$

Appendix B

We start from

$$\mathcal{H}_{2x} = U(x)\ \mathcal{H}_{1x} U^{-1}(x) \,, \qquad \forall x \in \mathcal{O}_1 \cap \mathcal{O}_2 \tag{B.1}$$

and we shall show that $\mathcal{H}_1 = \mathcal{H}_2$ up to an x-independent conjugation. We know from sect. 3 that almost everywhere in $\mathcal{O}_1 \cap \mathcal{O}_2$

$$\mathcal{H}_{ix} = \mathcal{S}_i \oplus \mathcal{A}_{ix} \,, \tag{B.2}$$

$$\mathcal{H}_i = \mathcal{S}_i \oplus \mathcal{A}_i \,, \quad i = 1.2 \,. \tag{B.3}$$

We then obtain for the semi-simple and Abelian parts separately

$$\mathcal{S}_2 = U(x)\ \mathcal{S}_1 U(x)^{-1}\,, \tag{B.4}$$

$$\mathcal{A}_{2x} = U(x)\ \mathcal{A}_{1x} U(x)^{-1}\,. \tag{B.5}$$

Now choose a point x_0 where (B.2) holds. We may now perform the constant gauge transformation $U^{-1}(x_0)$ in the patch $\mathcal{O}_2$ and obtain

$$\mathcal{H}'_2 = \mathcal{S}_1 \oplus \mathcal{A}'_2\,,$$

$$\mathcal{H}'_{2x} = \mathcal{S}_1 \oplus \mathcal{A}'_{2x}\,. \tag{B.6}$$

We can always take $\mathcal{A}_1$ and $\mathcal{A}'_2$ to be subalgebras of a standard form of the Cartan subalgebra $\mathcal{C}$ of $\mathcal{G}$, as discussed in sect. 3. It is convenient to consider a diagonal matrix representation of $\mathcal{C}$. From (B.5) we see that the action of $U(x)$ on $\mathcal{A}_{1x}$ is to transform diagonal matrices into diagonal matrices. This action can be performed by a constant, x-independent matrix K, which operates as a permutation on the eigenvalues of elements of $\mathcal{A}_{1x}$, and leaves $\mathcal{S}_1$ invariant:

$$\mathcal{A}'_{2x} = K\ \mathcal{A}_{1x} K^{-1}\,, \quad \forall x\,. \tag{B.7}$$

We now perform the transformation K^{-1} in the patch $\mathcal{O}_2$. After this transformation we have

$$\mathcal{A}''_{2x} = \mathcal{A}_{1x} \tag{B.8}$$

so that $\mathcal{H}_{1x} = \mathcal{H}''_{2x}$. By construction the smallest Abelian algebra containing $\mathcal{A}_{1x}$ and $\mathcal{A}''_{2x}$ are identical, and therefore $\mathcal{H}_1 = \mathcal{H}''_2$.

Appendix C

Let

$$\mathcal{H}_x = U(x)\ \mathcal{H}_x U(x)^{-1}\,. \tag{C.1}$$

We shall show that the most general form of $U(x)$ is

$$U(x) = U_{\mathrm{H}}(x)\, U_{\widetilde{\mathrm{H}}_\perp}(x)\, D\,, \tag{C.2}$$

where H and $\widetilde{\mathrm{H}}_\perp$ are the groups associated with $\mathcal{H}$ and $\widetilde{\mathcal{H}}_\perp$, and D is an x-independent operator, the properties of which will be given below. We know that

$$\mathcal{H}_x = \mathcal{S} \oplus \mathcal{A}_x\,, \tag{C.3}$$

and that

$$\mathcal{S} = U(x)\,\mathcal{S}\,U^{-1}(x)\,, \tag{C.4}$$

$$\mathcal{A}_x = U(x)\ \mathcal{A}_x U^{-1}(x)\,. \tag{C.5}$$

First, we establish *ab absurdo* that $U(x)$ acts as an automorphism of $\mathcal{H} \oplus \widetilde{\mathcal{H}}_\perp$. Let us assume that there exists an element a of $\mathcal{H} \oplus \widetilde{\mathcal{H}}_\perp$ such that

$$U(x)\, aU(x)^{-1} = b_{\mathcal{H}\oplus\widetilde{\mathcal{H}}_\perp} + b_\perp\,, \tag{C.6}$$

where $b_\perp \neq 0$ belongs to the orthogonal complement of $\mathcal{H} \oplus \widetilde{\mathcal{H}}_\perp$ in $\mathcal{G}$. We can assume that a has no component on $\mathcal{S}$, since from (C.4) we know that $U(x)$ acts as an automorphism of $\mathcal{S}$. Then we must have

$$[a,\ \mathcal{H}_x] = 0\,. \tag{C.7}$$

The action $U(x)$ on this last equation gives

$$[b_{\mathcal{H}\oplus\widetilde{\mathcal{H}}_\perp} + b_\perp,\ \mathcal{H}_x] = 0\,, \tag{C.8}$$

from which follows (see sect. 3)

$$[b_\perp,\ \mathcal{H}_x] = 0\,. \tag{C.9}$$

So $b_\perp$ belongs to $\widetilde{\mathcal{H}}_x = \widetilde{\mathcal{H}}$ (sect. 3), and we assume it to be orthogonal to $\widetilde{\mathcal{H}}$. So $b_\perp$ must vanish, and $U(x)$ acts as an automorphism of $\mathcal{H} \oplus \widetilde{\mathcal{H}}_\perp$. Any automorphism of $\mathcal{H} \oplus \widetilde{\mathcal{H}}_\perp$ can be written as the product of an inner automorphism by a discrete outer one [14]. So $U(x)$ can be written in the form (C.2).

It is a simple exercise to show from (C.1) and (C.2) that

$$D\mathcal{H}_x D^{-1} \subset \mathcal{H}\,. \tag{C.10}$$

References

[1] R. Jackiw, Rev. Mod. Phys. 49 (1977) 681.
[2] A. Belavin et al., Phys. Lett. 59B (1975) 85.
[3] R.G. Yates, Nucl. Phys. B123 (1977) 333.
[4] D.H. Tchrakian and J.H. Rawnsley, J. of Phys. A10 (1977) L195.
[5] Y. Brihaye and J. Nuyts, Phys. Lett. 66B (1977) 346.
[6] M.F. Atiyah, N.J. Hitchin and I.M. Singer, Proc. Nat. Acad. Sci. USA 74 (1977) 2662.
[7] H. Römer, CERN preprint TH.2409 (1977).
[8] E. Corrigan et al., Phys. Lett. 72B (1978) 354; Y. Brihaye et al., Mons and Durham preprint (1978).
[9] K.M. Bitar and P. Sorba, Phys. Rev. D16 (1977) 431.
[10] T.T. Wu and C.N. Yang, Phys. Rev. D12 (1975) 3843.
[11] C.H. Gu and C.N. Yang, Scientia Sinica 20 (1977) 47.
[12] C.W. Bernard et al., Phys. Rev. D16 (1977) 2967.
[13] R. Stora, Lectures given at Int. School of Math. Physics, Erice (1977).
[14] N. Jacobson, Lie Algebras (Interscience Publishers, New York, 1972).
[15] F. Wilczek, Phys. Lett. 65B (1976) 160.
[16] E. Witten, Phys. Rev. Lett. 38 (1977) 121.
[17] F.A. Bais and H.A. Weldon, University of Pennsylvania preprint 1978.